J. Fluid Mech. (1973), vol. 58, part 4, pp. 657–667
Printed in Great Britain

On the longitudinal dispersion of dye whose concentration varies harmonically with time

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(Received 29 September 1972)

In recent years many problems concerned with the dispersion of a passive contaminant along pipes and channels have been investigated, and this paper is concerned with one such problem which arises in diverse applications. This is the study of the longitudinal dispersion of a contaminant whose concentration is prescribed as a harmonic function of time at one cross-section. On the basis of physical arguments and of detailed calculations for two laminar flows it is shown that for high frequencies the concentration pattern is transported downstream at the maximum fluid velocity but that for low frequencies it is transported at the discharge velocity, and that the fluctuations in concentration decay to zero in a much shorter downstream distance for high frequencies than for low frequencies. It is shown further that at high frequencies the concentration is exponentially small except near the places where the fluid velocity attains its maximum, whereas for low frequencies the variation in concentration over the cross-section is small. Some of these conclusions are compared with those made by others, and the agreement is in general satisfactory.

1. Introduction

This paper analyses the way in which dye (or heat) diffuses in a pipe containing fluid in steady flow when the distribution of concentration (or temperature) is prescribed as a sinusoidal function of time at one fixed cross-section of the pipe. In particular, information about the velocity at which the pattern is transported downstream, and the rate at which it decays, is obtained for two laminar flows, and some remarks are made about turbulent flows.

Aspects of this problem have been investigated by three authors who were each motivated by different physical problems. Brinkman (1950) was concerned with the variation of temperature within a capillary tube so that he could investigate the variation of viscosity. Carrier (1956) investigated this problem because of its application to a group of experimental techniques in which the concentration of solute emerging from a long tube is determined and has to be related to the concentration at an upstream cross-section. One such technique is used in micrometeorology (with carbon dioxide and water vapour) and was mentioned as a motivation by Philip (1963*a*, *b*). Philip also referred to the dispersal of soluble materials in blood vessels and in the water-conducting organs of plants.

In addition it is evident that the mathematical problem described above is important in the investigation of any linear unsteady dispersion problem in a pipe because of the principle of superposition and the technique of Fourier analysis. That the problem is linear is equivalent to the assumptions that buoyancy effects are negligible† and that the molecular diffusion of dye (or heat) is independent of concentration (or temperature). Both of these assumptions will be made in this work so that attention is focused on the interaction between advection and diffusion.

The methods used in this paper differ from those used by the above authors. The results obtained here should, nevertheless, include those obtained by Philip, but there are differences, which will be discussed in §3.

2. Structure of the solution in laminar flow

Governing equations

Distance along the axis of the pipe will be measured by the co-ordinate x, with $x = 0$ being the cross-section at which the concentration C is prescribed as a sinusoidal function of time. In fact it will be assumed that at this place C is uniform over the cross-section so that, at $x = 0$,

$$C = C_0 + C_1 e^{i\omega t}, \tag{2.1}$$

where C_0 and C_1 are constants. The analysis could be extended to the more general case where C_0 and C_1 vary with position in the plane $x = 0$.

The pipe will be assumed to be straight and of constant cross-section. Thus in laminar flow the fluid velocity will be in the axial direction, steady, and dependent only on Y and Z, the co-ordinates in the cross-sectional plane made dimensionless by a length a typical of the cross-section. Suppose the discharge velocity is U. Then the fluid velocity can be written as $UV(Y,Z)$, and C satisfies the equation

$$\frac{\partial C}{\partial t} + UV(Y,Z)\frac{\partial C}{\partial x} = \kappa\frac{\partial^2 C}{\partial x^2} + \frac{\kappa}{a^2}\left(\frac{\partial^2 C}{\partial Y^2} + \frac{\partial^2 C}{\partial Z^2}\right), \tag{2.2}$$

where κ is the molecular diffusivity. Solutions of (2.2) must satisfy (2.1), and also a condition on the flux of C across the walls of the pipe. In this paper the case when the walls are impermeable (to matter or heat) will be the only one considered.‡ Thus on the boundary

$$\partial C/\partial n = 0. \tag{2.3}$$

Now (2.2) has solutions of the form

$$C = C_0 + \exp\left[i\omega(t - \lambda x/U)\right] f(Y,Z), \tag{2.4}$$

provided that
$$\frac{\partial^2 f}{\partial Y^2} + \frac{\partial^2 f}{\partial Z^2} = \frac{i\omega a^2}{\kappa}\left[1 - \lambda V(Y,Z)\right] f + \left(\frac{\omega a \lambda}{U}\right)^2 f. \tag{2.5}$$

The last term in (2.5) is the effect of axial diffusion and is of order $\omega\kappa\lambda^2/U^2$ times the first term on the right-hand side, which is the effect of advection. In

† This is a questionable assumption if concentrations are not low enough. See Erdogan & Chatwin (1967).

‡ The choice of (2.3) was made for two reasons: (*a*) because my main interest is in the dispersion of matter, and (*b*) to enable comparison to be made with the work of the authors referred to in § 1, all of whom used (2.3) (although Brinkman also considered the boundary condition C = constant).

the diffusion of matter in laminar flows in liquids $\kappa \approx 10^{-9}\,\mathrm{m^2\,s^{-1}}$, so that for the values $\omega \approx 2\pi\,(\mathrm{day})^{-1}$ and $U \approx 10^3\,\mathrm{m(day)^{-1}}$, which are typical for the experiments considered by Carrier (1956), $\omega\kappa/U^2 \approx 5{\cdot}5\times 10^{-10}$. For a case when $\omega \approx 2\pi\,\mathrm{s^{-1}}$ and $U \approx 10^{-2}\,\mathrm{m\,s^{-1}}$ then $\omega\kappa/U^2 \approx 6\times 10^{-5}$. For the diffusion of heat the values of $\omega\kappa/U^2$ are typically 100 times bigger. These numbers suggest that provided $\lambda = O(1)$, a result consistent with the results later in this paper, the effect of axial molecular diffusion can be ignored. Thus (2.5) will be replaced by

$$\frac{\partial^2 f}{\partial Y^2}+\frac{\partial^2 f}{\partial Z^2} = i\Omega[1-\lambda V(Y,Z)]f, \tag{2.6}$$

where

$$\Omega = \omega a^2/\kappa, \tag{2.7}$$

and the solutions of (2.6) must satisfy

$$\partial f/\partial n = 0 \tag{2.8}$$

on the boundary by virtue of (2.3).

Now (2.6) has solutions satisfying (2.8) only if λ is a member of the set of discrete eigenvalues, and for each possible λ there is one f determined up to a multiplicative constant. Now suppose f_p and f_q satisfy (2.6) and (2.8), with the values of λ equal to λ_p and λ_q respectively. By means of standard techniques it follows that, if $p \neq q$,

$$\iint V f_p f_q \,dY dZ = 0, \tag{2.9}$$

where the double integral is over the whole cross-section. Also, assuming the f_p form a complete set, then

$$C = C_0+\sum_p A_p \exp\,[i\omega(t-\lambda_p x/U)] f_p\,(Y,Z), \tag{2.10}$$

and the constants A_p can be determined so that (2.1) is satisfied, if the orthogonality relations (2.9) are used in the normal way.

Denote the complex conjugates of λ_p and f_p by overbars. From (2.6) it follows that

$$\frac{\partial^2 f_p}{\partial Y^2}+\frac{\partial^2 f_p}{\partial Z^2} = i\Omega\,[1-\lambda_p V] f_p,$$

$$\frac{\partial^2 \bar{f}_p}{\partial Y^2}+\frac{\partial^2 \bar{f}_p}{\partial Z^2} = -i\Omega\,[1-\bar{\lambda}_p V] \bar{f}_p,$$

and so, denoting the operator $(\partial/\partial Y, \partial/\partial Z)$ by ∇,

$$\nabla\,.\,[\bar{f}_p\nabla f_p - f_p\nabla\bar{f}_p] = 2i\Omega f_p\bar{f}_p\,[1-V\,\mathrm{Re}\,(\lambda_p)],$$

$$\nabla\,.\,[\bar{f}_p\nabla f_p + f_p\nabla\bar{f}_p] = 2\nabla f_p\,.\,\nabla\bar{f}_p + 2\Omega V f_p\bar{f}_p\,\mathrm{Im}\,(\lambda_p).$$

Integration of the last two equations over the cross-section gives, using (2.8),

$$\mathrm{Re}\,(\lambda_p) = \iint |f_p|^2 dY dZ \Big/ \iint V|f_p|^2 dY dZ \tag{2.11}$$

and

$$\mathrm{Im}\,(\lambda_p) = -(1/\Omega)\iint |\nabla f_p|^2 dY dZ \Big/ \iint V|f_p|^2 dY dZ. \tag{2.12}$$

It follows from (2.11) that, provided $V \geqslant 0$ everywhere,

$$1/V_{\max} \leqslant \mathrm{Re}\,(\lambda_p) < \infty, \tag{2.13}$$

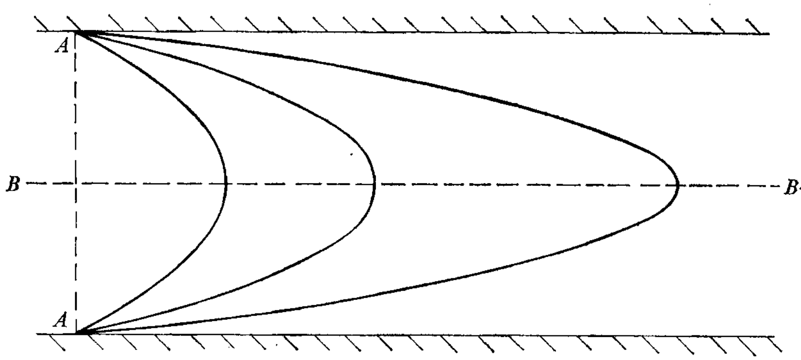

FIGURE 1. Sketch showing three equal-phase lines in Poiseuille flow (the curved lines). The particles on a given line were all at the cross-section AA (at which C is prescribed) at the same time, and the different equal-phase lines correspond to different times of 'release' from AA. The centre-line BB is that on which the fluid velocity attains its maximum. Notice that the equal-phase lines are closest together near the wall and furthest apart on BB.

where $V_{\max}$ is the maximum value attained by V. This result is not surprising. It says that the disturbance cannot be transported at a negative velocity nor at one greater than the maximum fluid velocity. But more information can be obtained. Suppose it happens that $|f_p|$ is small except near the place(s) where $V = V_{\max}$. Then the denominator in (2.11) is approximately $V_{\max}\iint |f_p|^2 dYdZ$ so that $\mathrm{Re}\,(\lambda_p)$ is approximately $1/V_{\max}$.

From (2.12), again supposing $V \geqslant 0$, it follows that

$$\mathrm{Im}\,(\lambda_p) < 0, \tag{2.14}$$

so that all solutions are spatially decaying. It is convenient to order the λ_p in ascending magnitude of their negative imaginary parts so that

$$0 < -\mathrm{Im}\,(\lambda_0) < -\mathrm{Im}\,(\lambda_1) < \ldots.$$

Hence, from (2.10), for large x,

$$C \approx C_0 + A_0 \exp\left[i\omega(t - \lambda_0 x/U)\right] f_0(Y, Z). \tag{2.15}$$

Thus for large x the concentration pattern is transported at a velocity $U/\mathrm{Re}\,(\lambda_0)$ and decays in an axial distance of order $-U/\mathrm{Im}\,(\lambda_0)$.

Expected results for high and low frequencies

Figure 1 shows, for one flow, lines of equal phase of concentration a short distance downstream of $x = 0$. In a short distance lateral diffusion has had little effect, so that the lines of equal phase are each similar to the velocity profile. In any flow the greatest lateral separation between two lines of equal phase is at and near the points where V has its maximum $V_{\max}$.

For a given phase difference two lines of equal phase will be closest together for high frequencies and the smoothing effect of lateral diffusion will then be most marked. This smoothing effect will be selective. Phase differences will persist longest near the line(s) where $V = V_{\max}$. It can therefore be anticipated that for high frequencies ($\Omega \gg 1$) the transport velocity will be near the maximum fluid velocity, that is that $\mathrm{Re}\,(\lambda_0)$ will be near $1/V_{\max}$.

For low frequencies ($\Omega \ll 1$) and a given phase difference two lines of equal phase are more widely separated than for high frequencies. The smoothing effect of lateral diffusion will therefore be less marked for $\Omega \ll 1$ than for $\Omega \gg 1$. Indeed the frequency may be so low that phase differences will persist for a long distance downstream, so that they become unimportant locally. Thus, following Taylor (1953), lines of equal phase will move downstream at the discharge velocity U, and will disperse longitudinally as a result of the local interaction between lateral diffusion and advection. The longitudinal dispersion will therefore be governed by a diffusion equation with a diffusion coefficient, K say, determined from the velocity distribution in the manner described by Taylor (1953). Hence, for $\Omega \ll 1$,

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = K\frac{\partial^2 C}{\partial x^2}, \tag{2.16}$$

and variations in C over the cross-section will be small compared with longitudinal variations. Thus $f_0(Y, Z)$ in (2.15) will be approximately constant; then (2.15) satisfies (2.16) provided

$$-K\omega\lambda_0^2/U^2 = i(1-\lambda_0). \tag{2.17}$$

But it is known (Taylor 1953) that

$$K = AU^2a^2/\kappa, \tag{2.18}$$

where A is a dimensionless number. From (2.17) and (2.18) it follows that, for $\Omega \ll 1$,

$$\lambda_0 = 1 - iA\Omega + O(\Omega^2). \tag{2.19}$$

3. Detailed calculations for Poiseuille flow in a circular pipe

Let a be the radius of the pipe, so that

$$V = 2(1-R^2), \quad \text{where} \quad R^2 = Y^2+Z^2. \tag{3.1}$$

Then (2.6) becomes

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{df}{dR}\right) = i\Omega[1-2\lambda(1-R^2)]f. \tag{3.2}$$

This equation can be reduced to the confluent hypergeometric equation (Philip 1963*a*), and has the everywhere-finite solution

$$f = \exp\left(-\tfrac{1}{2}\beta R^2\right) {}_1F_1(-\gamma;\ 1;\ \beta R^2), \tag{3.3}$$

where

$$\beta = (2i\lambda\Omega)^{\frac{1}{2}}, \quad \gamma = \left(\frac{i\Omega}{2\lambda}\right)^{\frac{1}{2}}\left(\frac{2\lambda-1}{4}\right) - \tfrac{1}{2}, \tag{3.4}$$

and the square roots have positive real parts. In (3.3), ${}_1F_1(a;\ b;\ z)$ is Kummer's function (Slater 1960, p. 2). Now

$$\frac{d}{dz}\,{}_1F_1(a;\ b;\ z) = \frac{a}{b}\,{}_1F_1(a+1;\ b+1;\ z),$$

so that (2.8) is satisfied if

$$ {}_1F_1(-\gamma;\, 1;\, \beta) + 2\gamma\, {}_1F_1(1-\gamma;\, 2;\, \beta) = 0. \tag{3.5} $$

The values of λ satisfying (3.5) are the required eigenvalues. Although (3.5) cannot be solved exactly, it is possible to derive approximate results for the cases $\Omega \gg 1$ and $\Omega \ll 1$ which were considered earlier.

High frequency input

For $\Omega \gg 1$ the real part of λ_0 is near $1/V_{\max} = \frac{1}{2}$, so that $|\beta| \gg 1$. Now for $|z| \to \infty$ and $\mathrm{Re}(z) > 0$

$$ {}_1F_1(a;\, b;\, z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \left[1 + O\left(\frac{1}{|z|}\right)\right] $$

(Slater 1960, p. 60). Hence (3.5) reduces to

$$ \frac{e^\beta}{\beta^{1+\gamma}\Gamma(-\gamma)} \left[1 + O\left(\frac{1}{|\beta|}\right)\right] = 0, $$

and this can be satisfied to highest order only if $\Gamma(-\gamma) \gg 1$, i.e. if

$$ \gamma \approx p\, (p = 0, 1, 2, \ldots). $$

Thus, using (3.4),

$$ \lambda_p \approx \tfrac{1}{2} + (2p+1)/(i\Omega)^{\frac{1}{2}} \quad (p = 0, 1, 2, \ldots), \tag{3.6} $$

and using (3.3), it follows that

$$ f_p \approx \exp\{-\tfrac{1}{2}(i\Omega)^{\frac{1}{2}} R^2\}\, {}_1F_1(-p;\, 1;\, (i\Omega)^{\frac{1}{2}} R^2). $$

In particular,

$$ f_0 \approx \exp\{-\tfrac{1}{2}(i\Omega)^{\frac{1}{2}} R^2\}. \tag{3.7} $$

Thus from (2.15), (3.6) and (3.7) it follows that, for $\Omega \gg 1$,

$$ C \approx C_0 + A_0 \exp\left[i\omega\left(t - \frac{x}{2U}\right)\right] \exp\left[-\left(\frac{i\omega\kappa}{U^2a^2}\right)^{\frac{1}{2}} x\right] \exp[-\tfrac{1}{2}(i\Omega)^{\frac{1}{2}} R^2], $$

provided x is not too small. So for high frequency the distribution of concentration is exponentially small except in a region of thickness $a\Omega^{-\frac{1}{4}}$ centred on the axis and decays in a distance of order $(Ua^2/\kappa)\,\Omega^{-\frac{1}{2}}$. The transport velocity is equal to $2U$, the maximum fluid velocity in the pipe. These conclusions accord with the qualitative arguments in §2.

Low frequency input

For $\Omega \ll 1$ the arguments of §2 suggest that λ_0 has a real part near 1 and an imaginary part of order Ω. This can be confirmed by supposing that (3.2) can be satisfied by expansions of the form (Carrier 1956):

$$ \left.\begin{aligned} \lambda &= \lambda_{00} + \lambda_{01}\Omega + \lambda_{02}\Omega^2 + \ldots, \\ f &= f_{00}(R) + f_{01}(R)\,\Omega + f_{02}(R)\,\Omega^2 + \ldots. \end{aligned}\right\} \tag{3.8} $$

Substitution into (3.2) and equating coefficients of like powers of Ω leads to an infinite set of differential equations which can be integrated only if the coefficients $\lambda_{00}, \lambda_{01}, \ldots$, in (3.8) have certain values. Substituting these values into the first equation of (3.8) gives

$$ \lambda_0 = 1 - \frac{i\Omega}{48} - \frac{\Omega^2}{1920} + O(\Omega^3). \tag{3.9} $$

	Transport velocity ($U/\mathrm{Re}(\lambda_0)$)	Decay distance ($-U/\omega \,\mathrm{Im}(\lambda_0)$)	Description of cross-sectional variation of C (determined from form of f_0)	Values of λ_p for $p \geqslant 1$
High frequency input: $\Omega \gg 1$	$2U$ (i.e. maximum fluid velocity)	$(Ua^2/\kappa)\,\Omega^{-\frac{1}{2}}$	Exponentially small except in a layer of thickness $a\Omega^{-\frac{1}{4}}$ centred on axis i.e. place at which fluid velocity is a maximum	$\lambda p = \frac{1}{2} + (2p+1)/(i\Omega)^{\frac{1}{2}}$ ($p = 1, 2, \ldots$)
Low frequency input: $\Omega \ll 1$	U (i.e. discharge velocity)	$48(Ua^2/\kappa)\,\Omega^{-2}$	Approximately uniform	$\lambda p = -\dfrac{8i}{9\Omega} \times (3p-2)^2$ ($p = 1, 2, \ldots$)

TABLE 1. Summary of results obtained for Poiseuille flow in a circular tube of radius a, discharge velocity U and molecular diffusivity κ. $\Omega = \omega a^2/\kappa$

For this value of λ_0, the value of f_0 is determined up to a constant of normalization. If this is chosen, without loss of generality, so that f_0 has unit mean over the cross-section then

$$f_0 = 1 + \tfrac{1}{24} i\Omega(1 - 6R^2 + 3R^4) + O(\Omega^2). \tag{3.10}$$

The details of the method are given by Carrier (1956) as are, in essence, the results (3.9) and (3.10). These results are consistent with (2.16), (2.18) and (2.19) since for this flow the value of A in (2.18) is $\frac{1}{48}$ (Taylor 1953).

The values of $\lambda_{00}, \lambda_{01}, \ldots$ are determined uniquely. Thus the values of λ_p for $p \gg 1$ cannot be expanded in power series like (3.8). Nor is it possible for λ_p to be of order Ω^n as $\Omega \to 0$ where $n > 0$ for this would imply a transport velocity of order $U\Omega^{-n}$ as $\Omega \to 0$, which is much greater than the maximum fluid velocity in the pipe and thus unacceptable. It is natural to suppose therefore that, for $p \gg 1$,

$$\lambda_p = X_p \Omega^{-n} e^{-i\theta_p},$$

where X_p, n and θ_p are positive, the last since $\mathrm{Im}(\lambda_p) < 0$, by (2.14). By means of asymptotic expansions of ${}_1F_1(a; b; z)$ for z and $2b - 4a$ both increasing in such a way that $z \approx (2b - 4a)$ (Slater 1960, pp. 86–88) it can be shown that $\theta_p = \frac{1}{2}\pi$, $n = 1$ and that

$$\lambda_p \approx -(8i/9\Omega)(3p-2)^2 \quad (p = 1, 2, \ldots). \tag{3.11}$$

As is usual in such applications of asymptotic expansions, the derivation of (3.11) is based on the assumption that p is large and so it cannot be expected to be very accurate for small p. However, it is significant that λ_p is purely imaginary for $p \gg 1$, so that only the component of C coming from λ_0 is *transported* away from $x = 0$. The other components simply decay.

The results of the calculations in this section are summarized in table 1. It will be noted that for given Ua^2/κ the concentration decays much more rapidly for $\Omega \gg 1$ than for $\Omega \ll 1$.

		λ_0	λ_1	λ_2
$\Omega = 10^2$	Present work: see (3.6)	$0 \cdot 57 - 0 \cdot 07i$	$0 \cdot 71 - 0 \cdot 21i$	$0 \cdot 85 - 0 \cdot 35i$
	Philip's results: see (3.13)	$0 \cdot 57 - 0 \cdot 08i$	$0 \cdot 73 - 0 \cdot 24i$	$2 \cdot 32 - 0 \cdot 32i$
$\Omega = 10^3$	Present work: see (3.6)	$0 \cdot 51 - 0 \cdot 01i$	$0 \cdot 52 - 0 \cdot 02i$	$0 \cdot 54 - 0 \cdot 04i$
	Philip's results: see (3.13)	$0 \cdot 53 - 0 \cdot 01i$	$0 \cdot 73 - 0 \cdot 02i$	$2 \cdot 36 - 0 \cdot 02i$

TABLE 2. Comparison of results obtained in this paper with those of Philip (1963*b*) for two large values of Ω

Comparison of results with those of others

Philip (1963*a*, *b*) investigated solutions of (3.2) by expanding $f(R)$ as a series of Bessel functions of order zero, viz.

$$f(R) = \sum_{m=0}^{\infty} B_m J_0(\mu_m R), \tag{3.12}$$

where $J_0'(\mu_m) = 0$ (so that (2.8) is satisfied) and $\mu_0 = 0$. Substitution of (3.12) into (3.2) leads after some algebra to an infinite determinantal equation for λ, which replaces equation (3.5) of the present paper. Philip assumed that the eigenvalues could be obtained approximately by replacing the infinite determinant by the finite determinants derived from it and containing successively 1, 2, 3, ..., terms of the leading diagonal. The determinant of order 1 gives a first approximation to λ_0; that of order 2 gives a second approximation to λ_0 and a first approximation to λ_1, and so on.

The adequacy of this method for $\Omega \gg 1$ is questionable since it can be shown that for $f_0(R)$ given by (3.7) the B_m in (3.12) increase with m for small m, so that f_0 cannot be well represented by the first few terms of (3.12). For $\Omega \gg 1$, Philip obtained

$$\left.\begin{aligned} \lambda_0 &= 0 \cdot 53 + \frac{11 \cdot 1}{i\Omega} + \frac{531}{\Omega^2} + O(\Omega^{-3}); \\ \lambda_1 &= 0 \cdot 73 + \frac{21 \cdot 7}{i\Omega} - \frac{769}{\Omega^2} + O(\Omega^{-3}); \\ \lambda_2 &= 2 \cdot 36 + \frac{23 \cdot 3}{i\Omega} + O(\Omega^{-2}). \end{aligned}\right\} \tag{3.13}$$

These results should be the same as (3.6). Table 2 compares (3.6) with (3.13) and it can be seen that the discrepancies are worse for $\Omega = 10^3$ than for $\Omega = 10^2$. This supports the preceding argument that Philip's method is incorrect as $\Omega \to \infty$.

For $\Omega \ll 1$, $f_p(R)$ is no longer concentrated near one value of R and the above criticisms of Philip's method for $\Omega \gg 1$ no longer apply. Philip obtained

$$\left.\begin{aligned} &\lambda_0 = 1 - 0 \cdot 0208 i\Omega - 0 \cdot 0005\Omega^2 + O(\Omega^3); \\ &\lambda_1 = \frac{12 \cdot 8}{i\Omega} + O(1); \quad \lambda_2 = \frac{51 \cdot 7}{i\Omega} + O(1). \end{aligned}\right\} \tag{3.14}$$

Brinkman (1950) obtained the values of

$$\lim_{\Omega\to 0}\{\lambda_p\,\Omega\}$$

for small p directly from the power-series expansion of ${}_1F_1(a;\, b;\, z)$. He obtained, effectively,

$$\lim_{\Omega\to 0}\{i\Omega\lambda_0\} = 0;\quad \lambda_1 = \frac{12\cdot 8}{i\Omega}+O(1);\quad \lambda_2 = \frac{41\cdot 9}{i\Omega}+O(1);\quad \lambda_3 = \frac{87\cdot 3}{i\Omega}+O(1). \tag{3.15}$$

The results in (3.14) and (3.15) should be consistent with those in (3.9) and (3.11). Since $\frac{1}{48} = 0\cdot 0208\ldots$ and $\frac{1}{1920} = 0\cdot 0005\ldots$, the value of λ_0 given by (3.9) is the same as that given by Philip. For λ_1, λ_2 and λ_3, (3.11) gives

$$\lambda_1 \approx 14\cdot 2/i\Omega;\quad \lambda_2 \approx 43\cdot 6/i\Omega;\quad \lambda_3 \approx 88\cdot 9/i\Omega. \tag{3.16}$$

The agreement between this and Brinkman's results is good, and best for λ_3. This is expected, since (3.11) was derived on the assumption that p is large. The value of λ_1 given by Philip agrees exactly with Brinkman's, but his value for λ_2 is about 25 % too high (although, as Philip notes, his value of λ_2 is only a *first* approximation).

4. Detailed calculations for Couette flow in a two-dimensional channel

Suppose that the channel has depth $2a$, and that the boundaries are at $Y = \pm 1$, with the one at $Y = -1$ at rest and that at $Y = +1$ moving. Then

$$V(Y) = (1+Y), \tag{4.1}$$

and (2.6) becomes

$$d^2f/dY^2 = i\Omega[1-\lambda-\lambda Y]f. \tag{4.2}$$

Now (4.2) has solution

$$f = M_1\,\mathrm{Ai}\left(\frac{\lambda Y+\lambda-1}{\alpha}\right)+M_2\,\mathrm{Bi}\left(\frac{\lambda Y+\lambda-1}{\alpha}\right), \tag{4.3}$$

where

$$\alpha = (-\lambda^2/i\Omega)^{\frac{1}{3}}, \tag{4.4}$$

the principal value of the cube root being taken, and Ai(z) and Bi(z) are Airy functions (Copson 1965, chap. 9). The boundary condition (2.8) can be satisfied for M_1 and M_2 not both zero only if

$$\mathrm{Ai}'\left(\frac{2\lambda-1}{\alpha}\right)\mathrm{Bi}'\left(-\frac{1}{\alpha}\right) = \mathrm{Ai}'\left(-\frac{1}{\alpha}\right)\mathrm{Bi}'\left(\frac{2\lambda-1}{\alpha}\right). \tag{4.5}$$

The values of λ which satisfy (4.5) are the required eigenvalues (so that (4.5) here plays the same role as (3.5) in Poiseuille flow).

For $\Omega \gg 1$ it is expected that Re(λ_p) is near $1/V_{\max} = \frac{1}{2}$. Write

$$2\lambda_p = 1+X_p\,\Omega^{-n}\,e^{-i\theta_p},$$

where X_p, n and θ_p are positive, the last since Im $(\lambda_p) < 0$, by (2.14). Thus, using (4.4),

$$\frac{1}{\alpha} \approx e^{-\frac{1}{6}i\pi}(4\Omega)^{\frac{1}{3}};\quad \left(\frac{2\lambda_p-1}{\alpha}\right) \approx e^{-i(\theta_p+\frac{1}{6}\pi)}\,X_p\,(4\Omega^{1-3n})^{\frac{1}{3}}.$$

It follows that $|1/\alpha| \gg 1$, and hence that

$$\mathrm{Ai}'(-1/\alpha) \approx -i\,\mathrm{Bi}'(-1/\alpha),$$

using the relevant asymptotic expansions. Hence (4.5) is equivalent to

$$\mathrm{Ai}'[X_p(4\Omega^{1-3n})^{\frac{1}{3}} e^{-i(\theta_p+\frac{1}{6}\pi)}] \approx -i\,\mathrm{Bi}'[X_p(4\Omega^{1-3n})^{\frac{1}{3}} e^{-i(\theta_p+\frac{1}{6}\pi)}],$$

and can be satisfied only if $n = \frac{1}{3}$. Then from asymptotic expansions for X_p large it follows after some algebra that

$$\lambda_p \approx \tfrac{1}{2} + \frac{1}{2}\left(\frac{3\pi}{16}\right)^{\frac{2}{3}} \left(\frac{1}{i\Omega}\right)^{\frac{1}{3}} (4p+1)^{\frac{2}{3}} \quad (p = 0, 1, 2, \ldots), \tag{4.6}$$

and thus that

$$f_p \approx \mathrm{Ai}\,[(1-Y)(\tfrac{1}{2}i\Omega)^{\frac{1}{3}} - (\tfrac{3}{8}\pi(4p+1))^{\frac{2}{3}}]. \tag{4.7}$$

Hence f_p is exponentially small except in a layer of thickness $a\Omega^{-\frac{1}{3}}$ near $Y = 1$, the place where the fluid velocity attains its maximum. From (4.6) it follows that the signal decays in a distance of order $(Ua^2/\kappa)\,\Omega^{-\frac{2}{3}}$. Apart from slight differences in the indices due to geometry these results are the same as those for Poiseuille flow.

For $\Omega \ll 1$ the techniques used for Poiseuille flow lead to the following results:

$$\lambda_0 = 1 - \frac{2i\Omega}{15} + O(\Omega^2); \quad \lambda_p = -\frac{9i\pi^2}{32\Omega}p^2 \quad (p = 1, 2, \ldots).$$

The value of λ_0 is consistent with the qualitative arguments in §2, since for this flow the value of Taylor's longitudinal diffusivity K is $\frac{2}{15}U^2a^2/\kappa$, so that A in (2.18) is $\frac{2}{15}$.

5. Some remarks on turbulent flow in a circular pipe

In laminar flow a^2/κ is the time taken for a fluid molecule to wander over the cross-section whereas the corresponding time in turbulent flow is a/u_*, where u_* is the friction velocity. Thus the parameter $\omega a/u_*$ plays the same role in turbulent flow as $\omega a^2/\kappa$ in laminar flow. The arguments of §2 then suggest that for $\omega a/u_* \gg 1$ the concentration pattern is transported downstream at the maximum fluid velocity (about $1{\cdot}1U$) and that it is observable only near the centre of the pipe. For $\omega a/u_* \ll 1$, on the other hand, it is expected that (2.16) holds with $K \approx 10{\cdot}1\, au_*$ (Taylor 1954), provided that the Reynolds number is high enough for the viscous sublayer to have negligible effect on the value of K. These conclusions receive some support from observations by Bentley & Dawson (1966) although these observations were made on the centre-line of a 1 in. diameter pipe in which there were bends and in which there were wire-wool obstructions. Figure 7 of this paper shows that for $\omega = 4\pi\,\mathrm{s}^{-1}$ the temperature pattern was transported at a velocity of the order of $1{\cdot}1U$ when the discharge velocity was of the order of $1\,\mathrm{ft\,s^{-1}}$, whereas the transport velocity was close to the discharge velocity when the latter was of the order of $10\,\mathrm{ft\,s^{-1}}$.

If it is assumed that the lateral mixing can be described by an eddy diffusivity calculated from Reynolds' analogy then quantitative estimates of the λ_p and f_p can be made, in principle at least. But there is the practical point that the fluctuating part of the distribution of concentration arising from the imposed variation will be difficult to separate from that due to the turbulent fluctuations.

I wish to thank Professor G. K. Batchelor for bringing this problem to my attention and for useful discussions. During the period when this work was done I was receiving financial support from the Science Research Council.

REFERENCES

Bentley, P. G. & Dawson, D. G. 1966 *Trans. Soc. Instrum. Tech.* **18**, 183.
Brinkman, H. C. 1950 *Appl. Sci. Res.* A**2**, 120.
Carrier, G. F. 1956 *Quart. Appl. Math.* **14**, 108.
Copson, E. T. 1965 *Asymptotic Expansions*. Cambridge University Press.
Erdogan, M. E. & Chatwin, P. C. 1967 *J. Fluid Mech.* **29**, 465.
Philip, J. R. 1963*a* *Aust. J. Phys.* **16**, 187.
Philip, J. R. 1963*b* *Aust. J. Phys.* **16**, 300.
Slater, L. J. 1960 *Confluent Hypergeometric Functions*. Cambridge University Press.
Taylor, G. I. 1953 *Proc. Roy. Soc.* A **219**, 186.
Taylor, G. I. 1954 *Proc. Roy. Soc.* A **223**, 446.